Direct factorization in formal concept analysis by factorization of input data

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Abstract: *Formal concept analysis aims at extracting a hierarchical structure (so-called concept lattice) of clusters (so-called formal concepts) from object-attribute data tables. We present an algorithm for computing a factor lattice of a concept lattice from the data and a user-specified similarity threshold a. The presented algorithm computes the factor lattice directly from the data, without first computing the whole concept lattice and then computing the collections of clusters. We present theoretical insight and examples.*

1 Problem Setting and Preliminaries

1.1 Problem Setting

Formal concept analysis (FCA) aims at extracting a hierarchical structure (so-called concept lattice) of clusters (so-called formal concepts) from object-attribute data tables. An important problem in applications of formal concept analysis is a possibly large number of clusters extracted from data. Factorization is one of the methods being used to cope with the number of clusters. We present an algorithm for computing a factor lattice of a concept lattice from the data and a user-specified similarity threshold a. The factor lattice is smaller than the original concept lattice and its size depends on the similarity threshold. The elements of the factor lattice are collections of clusters which are pairwise similar in degree at least a. The presented algorithm computes the factor lattice directly from the data, without first computing the whole concept lattice and then computing the collections of clusters. We present theoretical insight and examples for demonstration.

1.2 Preliminaries

For information on foundations and applications of FCA we refer to [5]. As to fuzzy logic, we refer to [8] and [6]. For information on FCA of data with fuzzy attributes that will be needed we refer to [1, 2, 3]. In what follows, we sumarize the basic notions.

Fuzzy logic We denote the scale of truth degrees by L. L together with logical connectives forms a structure $\mathbf{L}$ of truth degrees. We assume that $\mathbf{L}$ forms a so-called complete residuated lattice, i.e. $\otimes$ denotes fuzzy conjunction, $\rightarrow$ its corresponding (residuated) implication, and infima $\bigwedge$ and suprema $\bigvee$ exist in L. The most applied set of truth values is the real interval $[0, 1]$. Residuated lattices cover the most widely used logical operation like the minimum-based, Łukasiewicz-based, etc. The set of all fuzzy sets (or $\mathbf{L}$-sets) in X is denoted L^X. For fuzzy sets A, B in X we put $A \subseteq B$ (A is a subset of B) if for each $x \in X$ we have $A(x) \leq B(x)$.

Formal concept analysis of data with fuzzy attributes The basic notions are as follows. Let X and Y be sets of objects and attributes, respectively, $I : X \times Y \rightarrow L$ be a fuzzy relation between X and Y. $I(x, y)$ is the degree to which x has y. $\langle X, Y, I\rangle$ is called a formal fuzzy context (a data table with fuzzy attributes). For fuzzy sets $A \in L^X$ and $B \in L^Y$, define fuzzy sets $A^\uparrow \in L^Y$ and $B^\downarrow \in L^X$ by

$$A^\uparrow(y) = \bigwedge_{x\in X} (A(x) \to I(x, y)) \quad (1)$$

$$B^\downarrow(x) = \bigwedge_{y\in Y} (B(y) \to I(x, y)). \quad (2)$$

$A^\uparrow(y)$ is the truth degree of the fact "y is shared by all objects from A" and $B^\downarrow(x)$ is the truth degree of the fact "x has all attributes from B". Putting

$$\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \mid A^\uparrow = B,\ B^\downarrow = A\},$$

$\mathcal{B}(X, Y, I)$ is called a (fuzzy) concept lattice of $\langle X, Y, I\rangle$. Its elements $\langle A, B\rangle$, called formal concepts, are thought of as interesting clusters in data $\langle X, Y, I\rangle$ (they can be indeed interpreted as concepts in the sense of Port-Royal logic). Putting $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (iff $B_1 \supseteq B_2$), $\leq$ models the subconcept-superconcept hierarchy under which $\mathcal{B}(X, Y, I)$ is a complete lattice (see [2] for a further information and study of the structure of fuzzy concept lattices).

2 Fast factorization by similarity

2.1 Factorization by similarity

We first recall the parametrized method of factorization introduced in [1] to which we refer for details. Given a fuzzy context $\langle X, Y, I\rangle$, introduce a binary fuzzy relation $\approx$ on the set $\mathcal{B}(X, Y, I)$ of all formal concepts of $\langle X, Y, I\rangle$ by

$$(\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle) \ = \ \bigwedge_{x\in X} A_1(x) \leftrightarrow A_2(x)$$

for $\langle A_i, B_i\rangle \in \mathcal{B}(X, Y, I)$, $i = 1, 2$, where $\bigwedge$ is the infimum (minimum in most cases) and $\leftrightarrow$ is a connective of fuzzy equivalence defined by $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. $(\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle)$, called the degree of similarity of $\langle A_1, B_1\rangle$ and $\langle A_2, B_2\rangle$, is just the truth degree of "for each object x: x is covered by A_1 iff x is covered by A_2". It can be shown that $(\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle) = \bigwedge_{y\in Y} B_1(y) \leftrightarrow B_2(y)$. Therefore, measuring similarity of formal concepts via extents A_i coincides with measuring similarity via intents B_i, corresponnding to the duality of extent/intent view.

Given a truth degree $a \in L$ (a threshold specified by a user), consider the thresholded relation ${}^a\!\approx$ on $\mathcal{B}(X, Y, I)$ defined by $(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) \in {}^a\!\approx$ iff $(\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle) \geq a$. That is, ${}^a\!\approx$ is the relation "being similar in degree at least a". ${}^a\!\approx$ is reflexive and symmetric, but need not be transitive (it is transitive if $a \otimes b = a \wedge b$ holds true in $\mathbf{L}$, i.e. if we use minimum as fuzzy conjunction). Call a subset B of $\mathcal{B}(X, Y, I)$ a ${}^a\!\approx$-block if it is a maximal subset of $\mathcal{B}(X, Y, I)$ such that each two formal concepts from B are similar in degree at least a (the notion of a ${}^a\!\approx$-block generalizes that of an equivalence class: if ${}^a\!\approx$ is an equivalence relation, ${}^a\!\approx$-blocks are exactly the equivalence classes). Denote by $\mathcal{B}(X, Y, I)/{}^a\!\approx$ the collection of all ${}^a\!\approx$-blocks. It can be shown that ${}^a\!\approx$ are special intervals in the concept lattice $\mathcal{B}(X, Y, I)$. In detail, for a formal concept $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$, put

$$\langle A, B\rangle_a = \bigwedge\{\langle A', B'\rangle \mid (\langle A, B\rangle, \langle A', B'\rangle) \in {}^a\!\approx\},\ \langle A, B\rangle^a = \bigvee\{\langle A', B'\rangle \mid (\langle A, B\rangle, \langle A', B'\rangle) \in {}^a\!\approx\}.$$

That is $\langle A, B\rangle_a$ and $\langle A, B\rangle^a$ are the infimum and the supremum of the set of all formal concepts which are similar to $\langle A, B\rangle$ in degree at least a. Operators $\ldots_a$ and $\ldots^a$ are important in description of ${}^a\!\approx$-blocks:

Lemma 1 *${}^a\!\approx$-blocks are exactly intervals of $\mathcal{B}(X,Y,I)$ of the form $[\langle A,B\rangle_a, (\langle A,B\rangle_a)^a]$, i.e.*

$$\mathcal{B}(X,Y,I)/{}^a\!\approx \;=\; \{[\langle A,B\rangle_a, (\langle A,B\rangle_a)^a] \mid \langle A,B\rangle \in \mathcal{B}(X,Y,I)\}.$$

Note that an interval with lower bound $\langle A_1,B_1\rangle$ and upper bound $\langle A_2,B_2\rangle$ is the subset $[\langle A_1,B_1\rangle,\langle A_2,B_2\rangle] = \{\langle A,B\rangle \mid \langle A_1,B_1\rangle \le \langle A,B\rangle \le \langle A_2,B_2\rangle\}$. Now, define a partial order $\preceq$ on blocks of $\mathcal{B}(X,Y,I)/{}^a\!\approx$ by $[c_1,c_2] \preceq [d_1,d_2]$ iff $c_1 \le d_1$(iff $c_2 \le d_2$) where $[c_1,c_2],[d_1,d_2] \in \mathcal{B}(X,Y,I)/{}^a\!\approx$, i.e. c_1,c_2,d_1,d_2 are suitable formal concepts from $\mathcal{B}(X,Y,I)$ and $c_i \le d_i$ denotes that in $\mathcal{B}(X,Y,I)$, c_i is under (a subconcept of) d_i. Then we have

Theorem 1 *$\mathcal{B}(X,Y,I)/{}^a\!\approx$ equipped with $\preceq$ is a partially ordered set which is a complete lattice, the so-called factor lattice of $\mathcal{B}(X,Y,I)$ by similarity $\approx$ and a threshold a.*

Elements of $\mathcal{B}(X,Y,I)/{}^a\!\approx$ can be seen as similarity-based granules of formal concepts/clusters from $\mathcal{B}(X,Y,I)$. $\mathcal{B}(X,Y,I)/{}^a\!\approx$ thus provides a granular view on (the possibly large) $\mathcal{B}(X,Y,I)$. For further details and properties of $\mathcal{B}(X,Y,I)/{}^a\!\approx$ we refer to [1].

2.2 Similarity-based factorization of input data $\langle X,Y,I\rangle$ and direct computing of the factor lattice $\mathcal{B}(X,Y,I)/{}^a\!\approx$

Computing $\mathcal{B}(X,Y,I)/{}^a\!\approx$ using definition and Lemma 1, one has (1) to compute the whole concept lattice $\mathcal{B}(X,Y,I)$ and then (2) to compute ${}^a\!\approx$-blocks on $\mathcal{B}(X,Y,I)$, which can be quite demanding.

We are going to propose a way to compute $\mathcal{B}(X,Y,I)/{}^a\!\approx$ in a more efficient way: First, we propose a construction of a similarity-based factorization assigning to $\langle X,Y,I\rangle$ a "factorized data" $\langle X,Y,I\rangle/a$. Then we show that $\mathcal{B}(X,Y,I)/{}^a\!\approx$ is isomorphic to $\mathcal{B}(\langle X,Y,I\rangle/a)$. This reduces the computation of $\mathcal{B}(X,Y,I)/{}^a\!\approx$ to the computation of an ordinary fuzzy concept lattice $\mathcal{B}(\langle X,Y,I\rangle/a)$ for which we have an algorithm (see [3]) with a polynomial time delay complexity (see [7]).

We need some auxiliary results, for details we refer to [1, 2]. For a fuzzy set C in U and $a \in L$, the fuzzy sets $a \to C$ and $a \otimes C$ in U are defined by $(a \to C)(u) = a \to C(u)$ and $(a \otimes C)(u) = a \otimes C(u)$ for each $u \in U$. For fuzzy sets C,D in U, put $(C \approx D) = \bigwedge_{u\in U} C(u) \leftrightarrow D(u)$. Furthermore, we call a fuzzy set A in X an extent if there is a fuzzy set B in Y such that $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$ (similarly, B is an intent if there is A with $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$).

Lemma 2 *If A is an extent then so is $a \to A$; similarly, if B is an intent then so is $a \to B$.*

Proof. For extents (for intets, the argument is dual): The fact follows from the fact that extents are exactly the fixed point of a fuzzy closure operator ${}^{\uparrow\downarrow} : C \mapsto C^{\uparrow\downarrow}$ and the fact that for a fixed point A of a fuzzy closure operator, $a \to A$ is a fixed point as well. □

The next lemma shows that for a formal concept $\langle A,B\rangle$, $\langle A,B\rangle_a$ and $\langle A,B\rangle^a$ (defined as infimum and supremum of all formal concepts similar to $\langle A,B\rangle$ in degree at least a) can be computed from $\langle A,B\rangle$ directly.

Lemma 3 *For $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$, we have (a) $\langle A,B\rangle_a = \langle (a\otimes A)^{\uparrow\downarrow}, a \to B\rangle$ and (b) $\langle A,B\rangle^a = \langle (a \to A), (a \otimes B)^{\downarrow\uparrow}\rangle$.*

Proof. We give only a sketch of (a). First, one can show that (a1) $(a\otimes A)^{\uparrow\downarrow}$ is an extent of a formal concept $\langle (a\otimes A)^{\uparrow\downarrow}, D\rangle$ which is similar to $\langle A,B\rangle$ in degree at least a. Second, one can show that (a2) if $\langle C,F\rangle$ is a formal concept similar to $\langle A,B\rangle$ in degree at least a then $\langle (a\otimes A)^{\uparrow\downarrow}, D\rangle \le \langle C,F\rangle$. From (a1) and (a2) we have that $\langle (a\otimes A)^{\uparrow\downarrow}, D\rangle$ is the least formal concept similar to $\langle A, B\rangle$ in degree at least a. Therefore, $\langle A, B\rangle_a = \langle (a\otimes A)^{\uparrow\downarrow}, D\rangle$. It remains to show $D = a \rightarrow B$. This can be done by showing that (a3) $a \rightarrow B$ is an intent of a concept c which is similar to $\langle A, B\rangle$ in degree at least a, and (a4) if $\langle C, F\rangle$ is a concept similar to $\langle A, B\rangle$ in degree at least a then $c \leq \langle C, F\rangle$. Indeed, from (a3) and (a4) it follows that $a \rightarrow B$ is the intent of the least formal concept similar to $\langle A, B\rangle$ in degree at least a, i.e. $a \rightarrow B = D$. Verification of (a3) and (a4) completes the proof. $\square$

Therefore, it follows that $(\langle A, B\rangle_a)^a = \langle a \rightarrow (a\otimes A)^{\uparrow\downarrow}, (a\otimes(a \rightarrow B))^{\downarrow\uparrow}\rangle$.

For a formal fuzzy context $\langle X, Y, I\rangle$ and a (user-specified) threshold $a \in L$, introduce a formal fuzzy context $\langle X, Y, I\rangle/a$ by

$$\langle X, Y, I\rangle/a \;:=\; \langle X, Y, a \rightarrow I\rangle.$$

$\langle X, Y, I\rangle/a$ will be called the factorized context of $\langle X, Y, I\rangle$ by threshold a. That is, $\langle X, Y, I\rangle/a$ has the same objects and attributes as $\langle X, Y, I\rangle$, and the incidence relation of $\langle X, Y, I\rangle/a$ is $a \rightarrow I$. Since

$$(a \rightarrow I)(x, y) \;=\; a \rightarrow I(x, y),$$

computing $\langle X, Y, I\rangle/a$ from $\langle X, Y, I\rangle$ is easy. The following is our main theorem.

Theorem 2 *For a formal fuzzy context $\langle X, Y, I\rangle$ and a threshold $a \in L$ we have*

$$\mathcal{B}(X, Y, I)/^a\!\approx\; \cong\; \mathcal{B}(\langle X, Y, I\rangle/a).$$

In words, $\mathcal{B}(X, Y, I)/^a\!\approx$ is isomorphic to $\mathcal{B}(\langle X, Y, I\rangle/a)$. Moreover, under the isomorphism, $[\langle A_1, B_1\rangle, \langle A_2, B_2\rangle] \in \mathcal{B}(X, Y, I)/^a\!\approx$ corresponds to $\langle A_2, B_1\rangle \in \mathcal{B}(\langle X, Y, I\rangle/a)$.

Proof. Let $^\uparrow$ and $^\downarrow$ denote the operators induced by I and $^{\uparrow_a}$ and $^{\downarrow_a}$ denote the operators induced by $a \rightarrow I$. Take any $A \in L^X$. Then $A^{\uparrow_a}(y) = \bigwedge_{x\in X} A(x) \rightarrow (a \rightarrow I(x, y)) = \bigwedge_{x\in X} a \rightarrow (A(x) \rightarrow I(x, y)) = a \rightarrow \bigwedge_{x\in X}(A(x) \rightarrow I(x, y)) = a \rightarrow A^\uparrow(x)$, and $A^{\uparrow_a\downarrow_a}(x) = \bigwedge_{y\in Y} A^{\uparrow_a}(y) \rightarrow (a \rightarrow I(x, y)) = \bigwedge_{y\in Y} a \rightarrow (A^{\uparrow_a}(y) \rightarrow I(x, y)) = a \rightarrow \bigwedge_{y\in Y}(A^{\uparrow_a}(y) \rightarrow I(x, y)) = a \rightarrow \bigwedge_{y\in Y}([\bigwedge_{x\in X} a \rightarrow (A(x) \rightarrow I(x, y))] \rightarrow I(x, y)) = a \rightarrow \bigwedge_{y\in Y}([\bigwedge_{x\in X}(a\otimes A(x)) \rightarrow I(x, y)] \rightarrow I(x, y)) = a \rightarrow \bigwedge_{y\in Y}((a\otimes A)^\uparrow(x) \rightarrow I(x, y)) = a \rightarrow (a\otimes A)^{\uparrow\downarrow}(x)$, i.e.

$$A^{\uparrow_a} = a \rightarrow A^\uparrow \quad \text{and} \quad A^{\uparrow_a\downarrow_a} = a \rightarrow (a\otimes A)^{\uparrow\downarrow}. \tag{3}$$

Now, let $[\langle A_1, B_1\rangle, \langle A_2, B_2\rangle] \in \mathcal{B}(X, Y, I)/^a\!\approx$. Using Lemma 1 and Lemma 3, there is $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ such that $\langle A_1, B_1\rangle = \langle A, B\rangle_a = \langle (a\otimes A)^{\uparrow\downarrow}, a \rightarrow B\rangle$ and $\langle A_2, B_2\rangle = (\langle A, B\rangle_a)^a = \langle a \rightarrow (a\otimes A)^{\uparrow\downarrow}, (a\otimes(a \rightarrow B))^{\downarrow\uparrow}\rangle$. Since $\langle A, B\rangle = \langle A, A^\uparrow\rangle$, (3) yields $A_2 = a \rightarrow (a\otimes A)^{\uparrow\downarrow} = A^{\uparrow_a\downarrow_a}$ and $B_1 = a \rightarrow B = a \rightarrow A^\uparrow = A^{\uparrow_a}$. This shows $\langle A_2, B_1\rangle \in \mathcal{B}(X, Y, a \rightarrow I) = \mathcal{B}(\langle X, Y, I\rangle/a)$.

Conversely, if $\langle A_2, B_1\rangle \in \mathcal{B}(\langle X, Y, I\rangle/a)$ then using (3), $B_1 = A_2^{\uparrow_a} = a \rightarrow A_2^\uparrow$ and $A_2 = A_2^{\uparrow_a\downarrow_a} = a \rightarrow (a\otimes A_2)^{\uparrow\downarrow}$. By Lemma 1 and Lemma 3, $[\langle B_1^\downarrow, B_1\rangle, \langle A_2, A_2^\uparrow\rangle] \in \mathcal{B}(X, Y, I)/^a\!\approx$. The proof is complete. $\square$

Remark 1. (1) As we have seen, the blocks of $\mathcal{B}(X, Y, I)/^a\!\approx$ can be reconstructed from the formal concepts of $\mathcal{B}(\langle X, Y, I\rangle/a)$: If $\langle A, B\rangle \in \mathcal{B}(\langle X, Y, I\rangle/a)$ then $[\langle B^\downarrow, B\rangle, \langle A, A^\uparrow\rangle] \in \mathcal{B}(X, Y, I)/^a\!\approx$.

(2) Computing $\mathcal{B}(\langle X, Y, I\rangle/a)$ means computing of the ordinary fuzzy concept lattice. This can be done by an algorithm of polynomial time delay complexity, see [3].

This shows a way to obtain $\mathcal{B}(X, Y, I)/^a\!\approx$ without computing first the whole $\mathcal{B}(X, Y, I)$ and then computing the factorization. Note that in [4], we showed an alternative way to speed up the computation of $\mathcal{B}(X, Y, I)/^a\!\approx$ by showing that suprema of blocks of $\mathcal{B}(X, Y, I)/^a\!\approx$ are

Table 1: Data table (fuzzy context).

	1	2	3	4	5	6	7
1 Czech	0.4	0.4	0.6	0.2	0.2	0.4	0.2
2 Hungary	0.4	1.0	0.4	0.0	0.0	0.4	0.2
3 Poland	0.2	1.0	1.0	0.0	0.0	0.0	0.0
4 Slovakia	0.2	0.6	1.0	0.0	0.2	0.2	0.2
5 Austria	1.0	0.0	0.2	0.2	0.2	1.0	1.0
6 France	1.0	0.0	0.6	0.4	0.4	0.6	0.6
7 Italy	1.0	0.2	0.6	0.0	0.2	0.6	0.4
8 Geramny	1.0	0.0	0.6	0.2	0.2	1.0	0.6
9 UK	1.0	0.2	0.4	0.0	0.2	0.6	0.6
10 Japan	1.0	0.0	0.4	0.2	0.2	0.4	0.2
11 Canada	1.0	0.2	0.4	1.0	1.0	1.0	1.0
12 USA	1.0	0.2	0.4	1.0	1.0	0.2	0.4

attributes: 1 - Gross Domestic Product per capita (USD), 2 - Consumer Price Index (1995=100), 3 - Unemployment Rate (percent - ILO), 4 - Production of electricity per capita (kWh), 5 - Energy consumption per capita (GJ), 6 - Export per capita (USD), 7 - Import per capita (USD)

fixed points of a certain fuzzy closure operator. Compared to that, the present approach shows that the blocks of $\mathcal{B}(X,Y,I)/^a\!\approx$ can be interpreted as formal concepts in a "factorized context" $\langle X,Y,I\rangle/a$, i.e. in a context in which objects and attributes are more similar than in the original context $\langle X,Y,I\rangle$. Indeed, for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ one can easily verify that

$$I(x_1,y_1) \leftrightarrow I(x_1,y_1) \leq (a \to I)(x_1,y_1) \leftrightarrow (a \to I)(x_2,y_2)$$

which intuitively says that in the factorized context, the table entries are more similar (closer) than in the original one.

3 Examples and experiments

Due to the limited scope, we demonstrate our algorithm on a data table (fuzzy context) from Tab. 1 for which we consider various parameters a (threshold) and some characteristics for comparison. The data table contains countries (objects from X) and some of their economic characteristics (attributes from Y). The original values of the characteristics are scaled to interval $[0,1]$ so that the characteristics can be considered as fuzzy attributes. Tab. 2 summarizes the effect of our algorithm and some related characteristics when using Łukasiewicz fuzzy logical connectives. The whole concept lattice $\mathcal{B}(X,Y,I)$ contains 774 formal concepts, computing $\mathcal{B}(X,Y,I)$ using the polynomial time delay algorithm from [3] takes 2292ms. The columns correspond to different threshold values $a = 0.2, 0.4, 0.6, 0.8$. Entries "size $|\mathcal{B}(X,Y,I)/^a\!\approx|$" contain the number of $^a\!\approx$-blocks; "naive algorithm (ms)" contain the time in ms for computing $\mathcal{B}(X,Y,I)/^a\!\approx$ by first generating $\mathcal{B}(X,Y,I)$ and subsequently generating the $^a\!\approx$-blocks by producing $[\langle A,B\rangle_a, (\langle A,B\rangle_a)^a]$; "our algorithm (ms)" contain the time in ms for computing $\mathcal{B}(X,Y,I)/^a\!\approx$ by reduction to the computation of $\mathcal{B}(\langle X,Y,I\rangle/a)$; "reduction $|\mathcal{B}(X,Y,I)/^a\!\approx|/|\mathcal{B}(X,Y,I)|$" contain the reduction factors of the size of the concept lattice; "time reduction" contain "our algorithm (ms)" divided by "naive algorithm (ms)" (1/"time reduction" is thus the speedup). Tab. 3 shows the same characteriztics when using the minimum-based fuzzy logical operations.

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Table 2: Łukasiewicz fuzzy logical connectives, $\mathcal{B}(X,Y,I)$ of data from Tab. 1: $|\mathcal{B}(X,Y,I)| = 774$, time for computing $\mathcal{B}(X,Y,I) = 2292$ ms; table entries for thresholds $a = 0.2, 0.4, 0.6, 0.8$.

	0.2	0.4	0.6	0.8
size $\lvert\mathcal{B}(X,Y,I)/^a\!\approx\rvert$	8	57	193	423
naive algorithm (ms)	8995	9463	8573	9646
our algorithm (ms)	23	214	383	1517
reduction $\lvert\mathcal{B}(X,Y,I)/^a\!\approx\rvert/\lvert\mathcal{B}(X,Y,I)\rvert$	0.010	0.073	0.249	0.546
time reduction	0.002	0.022	0.044	0.157

Table 3: Minimum-based fuzzy logical connectives, $\mathcal{B}(X,Y,I)$ of data from Tab. 1: $|\mathcal{B}(X,Y,I)| = 304$, time for computing $\mathcal{B}(X,Y,I) = 341$ ms; table entries for thresholds $a = 0.2, 0.4, 0.6, 0.8$.

	0.2	0.4	0.6	0.8
size $\lvert\mathcal{B}(X,Y,I)/^a\!\approx\rvert$	8	64	194	304
naive algorithm (ms)	1830	1634	3787	4440
our algorithm (ms)	23	106	431	1568
reduction $\lvert\mathcal{B}(X,Y,I)/^a\!\approx\rvert/\lvert\mathcal{B}(X,Y,I)\rvert$	0.026	0.210	0.638	1.000
time reduction	0.012	0.064	0.113	0.353

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