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Fast factorization by similarity in formal concept analysis of data with fuzzy attributes

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Abstract

We present a method of fast factorization in formal concept analysis (FCA) of data with fuzzy attributes. The output of FCA consists of a partially ordered collection of clusters extracted from a data table describing objects and their attributes. The collection is called a concept lattice. Factorization by similarity enables us to obtain, instead of a possibly large concept lattice, its factor lattice. The elements of the factor lattice are maximal blocks of clusters which are pairwise similar to degree exceeding a user-specified threshold. The factor lattice thus represents an approximate version of the original concept lattice. We describe a fuzzy closure operator the fixed points of which are just clusters which uniquely determine the blocks of clusters of the factor lattice. This enables us to compute the factor lattice directly from the data without the need to compute the whole concept lattice. We present theoretical solution and examples demonstrating the speed-up of our method.

Keywords: Tabular data; Clustering; Formal concept analysis; Fuzzy attributes; Similarity; Factorization

1. Problem setting

Finding interesting and well-interpretable groups in data is a challenging goal. Formal concept analysis (FCA) is a method of exploratory data analysis which aims at extracting a hierarchical structure of clusters from tabular data describing objects and their attributes [9]. The history of FCA goes back to Wille's paper [15], foundations, algorithms, and a survey of applications can be found in [7]. Recently, there appeared interesting applications of FCA as a data analysis method, see e.g. [1,8,12,13], as well as a data preprocessing method, see e.g. [14,18].

Clusters in FCA are particular pairs $\langle A, B\rangle$ consisting of a collection A of objects and a collection B of attributes which are maximal with respect to the property that each object from A has every attribute from B. Since this approach corresponds to Port–Royal idea of a concept consisting of its extent (objects covered by the concept) and its intent (attributes covered by the concept), clusters $\langle A, B\rangle$ are called formal concepts. Formal concepts can be partially ordered by a subconcept–superconcept hierarchy (narrower clusters are under larger ones). The resulting partially ordered set of clusters forms a complete lattice, called a concept lattice, and can be visualized by a labeled Hasse dia-

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gram. In basic setting, the input data table contains bivalent attributes, i.e. each table entry contains either 0 or 1. More general attributes are handled by so-called conceptual scaling [9]. Recently, FCA was extended to data tables with fuzzy attributes, i.e. tables with entries containing degrees to which a particular attribute applies to a particular object. Then, the constituents A and B of a formal concept $\langle A, B\rangle$ are fuzzy sets rather than bivalent sets, see e.g. [4,5,11].

In [2], a method of parameterized factorization of concept lattices computed from data tables with fuzzy attributes is presented. A user supplies a similarity threshold a (parameter) and the method outputs, instead of the whole concept lattice which might be large, its factor lattice. The elements of the factor lattice are maximal blocks of clusters from the whole concept lattice which are pairwise similar to degree at least a. For a user, the factor lattice provides a coarser version of the whole concept lattice—the less the similarity threshold a, the coarser. In order to compute the factor lattice directly by definition, we have to compute the whole concept lattice (this can be done by an algorithm with a polynomial time delay, see [6]) and then compute all the similarity blocks, i.e. elements of the factor lattice (again, this can be accomplished by an algorithm with polynomial time delay). In this paper, we present a way to compute the factor lattice directly from input data. The resulting algorithm is significantly faster than computing first the whole concept lattice and then computing the similarity blocks. In addition to that, the smaller the similarity threshold, the faster the computation of the factor lattice. This feature corresponds to a rule "the more tolerance to imprecision, the faster the result" which is characteristic for human categorization [16,17].

The paper is organized as follows. Section 2 presents preliminaries on fuzzy sets and formal concept analysis of data with fuzzy attributes. In Section 3 we present our approach. Section 4 presents experiments and demonstrates the speed-up.

2. Preliminaries

2.1. Fuzzy sets and fuzzy logic

The concept of a fuzzy set generalizes that of an ordinary set in that an element may belong to a fuzzy set in an intermediate degree not necessarily being 0 or 1. Formally, a fuzzy set A in a universe X [10] is a mapping assigning to each $x \in X$ a truth degree $A(x) \in L$ where L is some partially ordered set of truth degrees containing at least 0 (false) and 1 (true). Usually, L is the unit interval $[0, 1]$ or some of its subsets. $A(x)$ is interpreted as a degree to which x belongs to A. The notion of a fuzzy set enables us to model vaguely (nonsharply) delineated collections: For instance, the collection corresponding to "tall man" can be modeled by a fuzzy set to which men with heights 150 cm, 180 cm, and 200 cm belong to degrees 0, 0.7, and 1, respectively.

The scale L of truth degrees needs to be equipped with suitable operations generalizing logical connectives of classical (bivalent) logic. Particularly, we will need fuzzy conjunction $\otimes$ and fuzzy implication $\to$. We assume that the set of truth degrees and the logical connectives form a complete residuated lattice $\mathbf{L}$ [2], i.e. $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$, where (1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice (with the least element 0, greatest element 1), i.e. a partially ordered set in which arbitrary infima ($\bigwedge$, for semantics of general quantifier) and suprema ($\bigvee$, for semantics of existential quantifier) exist; (2) $\otimes$ satisfies $x \otimes (y \otimes z) = (x \otimes y) \otimes z$, $x \otimes y = y \otimes x$, and $x \otimes 1 = x$; (3) $\otimes$ and $\to$ satisfy $x \otimes y \leqslant z$ if and only if $x \leqslant y \to z$ (adjointness, comes from *modus ponens*). $\otimes$ and $\to$ are called multiplication and residuum and play the role of fuzzy conjunction and fuzzy implication, respectively. The most commonly used set L of truth degrees is the real interval $[0, 1]$; with $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and with three important pairs of fuzzy conjunction and fuzzy implication: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \to b = \min(1 - a + b, 1)$), minimum ($a \otimes b = \min(a, b)$, $a \to b = 1$ if $a \leqslant b$ and $= b$ else), and product ($a \otimes b = a \cdot b$, $a \to b = 1$ if $a \leqslant b$ and $= b/a$ else). Another possibility is to take a finite chain $\{a_0 = 0, a_1, \dots, a_n = 1\}$ ($a_0 < \dots < a_n$) equipped with Łukasiewicz structure ($a_k \otimes a_l = a_{\max(k+l-n,0)}$, $a_k \to a_l = a_{\min(n-k+l,n)}$) or minimum ($a_k \otimes a_l = a_{\min(k,l)}$, $a_k \to a_l = a_n$ for $a_k \leqslant a_l$ and $a_k \to a_l = a_l$ otherwise). More generally, taking $I = \{i_1 = 0, \dots, i_m = n\} \subseteq \{0, \dots, n\}$ with $i_0 < \dots < i_m$, one can define a pair of a fuzzy conjunction and a fuzzy implication by

$$a_k \otimes a_l = \begin{cases} a_{\max(k+l-i_{j+1}, i_j)} & \text{if } k, l \in [i_j, i_{j+1}], \\ a_{\min(k,l)} & \text{otherwise}, \end{cases}$$

$$a_k \to a_l = \begin{cases} 1 & \text{if } k \leqslant l, \\ a_{\min(i_{j+1}-k+l, i_{j+1})} & \text{if } k > l \text{ and } k, l \in [i_j, i_{j+1}], \\ a_l & \text{otherwise}. \end{cases}$$

One can see that for $I = \{0, 1\}$ and $I = \{0, 1, \dots, n\}$ we get the above-mentioned Łukasiewicz and minimum pair, respectively.

A fuzzy set with truth degrees from $\mathbf{L}$ in a universe U (called also an $\mathbf{L}$-set) is a mapping $A : U \to L$ assigning to any $u \in U$ a truth degree $A(u) \in L$ to which u belongs to A. A fuzzy relation I between sets X and Y is a fuzzy set in $U = X \times Y$, i.e. $I : X \times Y \to L$. The set of all $\mathbf{L}$-sets in a universe U is denoted by L^U. For a fuzzy set $A \in L^U$ and a truth degree $a \in L$ we denote by ${}^aA$ the a-cut of A, i.e. ${}^aA = \{u \in U \mid A(u) \geqslant a\}$ (the ordinary set of elements from U which belong to A to degree at least a). A fuzzy set A is called crisp if $A(u) \in \{0, 1\}$. Following common usage we will identify crisp fuzzy sets in U with ordinary subsets of U. For fuzzy sets A, B in U we put $A \subseteq B$ (A is a subset of B) if for each $u \in U$ we have $A(u) \leqslant B(u)$. More generally, the degree $S(A, B)$ to which A is a subset of B is defined by $S(A, B) = \bigwedge_{u \in U} A(u) \to B(u)$. Then, $A \subseteq B$ means $S(A, B) = 1$.

2.2. Formal concept analysis of data with fuzzy attributes

Let X and Y be sets of objects and attributes, respectively, I be an $\mathbf{L}$-relation between X and Y, i.e. I is a mapping $I : X \times Y \to L$. $\langle X, Y, I\rangle$ is called a data table with fuzzy attributes. $\langle X, Y, I\rangle$ represents a table which assigns to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ to which object x has attribute y.

For $\mathbf{L}$-sets $A \in L^X$, $B \in L^Y$ (i.e. A is an $\mathbf{L}$-set of objects, B is an $\mathbf{L}$-set of attributes), we define $\mathbf{L}$-sets $A^{\uparrow} \in L^Y$ ($\mathbf{L}$-set of attributes), $B^{\downarrow} \in L^X$ ($\mathbf{L}$-set of objects) by

$$A^{\uparrow}(y) = \bigwedge_{x \in X} \bigl(A(x) \to I(x, y)\bigr) \quad \text{and} \quad B^{\downarrow}(x) = \bigwedge_{y \in Y} \bigl(B(y) \to I(x, y)\bigr).$$

We put

$$\mathcal{B}(X, Y, I) = \bigl\{\langle A, B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \bigm| A^{\uparrow} = B,\ B^{\downarrow} = A\bigr\}$$

and define for $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X, Y, I)$ a binary relation $\leqslant$ by $\langle A_1, B_1\rangle \leqslant \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (or, iff $B_2 \subseteq B_1$; both ways are equivalent). Operators ${}^{\downarrow}$, ${}^{\uparrow}$ induced by $\langle X, Y, I\rangle$ form a fuzzy Galois connection [4]. The structure $\langle \mathcal{B}(X, Y, I), \leqslant\rangle$ is called a concept lattice induced by $\langle X, Y, I\rangle$. Elements $\langle A, B\rangle$ of $\mathcal{B}(X, Y, I)$ are called formal concepts and are interpreted as concepts/clusters hidden in the input data table. Namely, $A^{\uparrow} = B$ and $B^{\downarrow} = A$ say that B is the collection of all attributes shared by all objects from A, and A is the collection of all objects sharing all attributes from B. Note that these conditions represent exactly the definition of a concept as developed in Port–Royal logic; A and B are called extent and intent of the concept $\langle A, B\rangle$, respectively, and represent the collection of all objects and all attributes covered by the particular concept. Furthermore, $\leqslant$ is the subconcept–superconcept hierarchy—concept $\langle A_1, B_1\rangle$ is a subconcept of $\langle A_2, B_2\rangle$ iff each object from A_1 belongs to A_2 (dually for attributes). $\langle \mathcal{B}(X, Y, I), \leqslant\rangle$ is a complete lattice, see [5] for more information about its structure.

3. Fast factorization by similarity

3.1. Factorization by similarity

In this section, we recall the parametrized method of factorization introduced in [2]. For details, we refer to [2] and to [9] where a general factorization of complete lattices by tolerance relations is described. Given a data table $\langle X, Y, I\rangle$, introduce a binary fuzzy relation $\approx$ on the set $\mathcal{B}(X, Y, I)$ of all formal concepts of $\langle X, Y, I\rangle$ by

$$\bigl(\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle\bigr) = \bigwedge_{x \in X} A_1(x) \leftrightarrow A_2(x)$$

for $\langle A_i, B_i\rangle \in \mathcal{B}(X, Y, I)$, $i = 1, 2$. Here, $\bigwedge$ denotes infimum and $\leftrightarrow$ is a connective of fuzzy equivalence defined by $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. It is known that $\approx$ is a fuzzy equivalence relation, i.e. we have $(A \approx A) = 1$ (reflexivity), $(A_1 \approx A_2) = (A_2 \approx A_1)$ (symmetry), and $(A_1 \approx A_2) \otimes (A_2 \approx A_3) \leqslant (A_1 \approx A_3)$ (transitivity). $(\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle)$ is called the degree of similarity of $\langle A_1, B_1\rangle$ and $\langle A_2, B_2\rangle$. It is easily seen that it is the truth degree of "for each object x: x is covered by A_1 iff x is covered by A_2." One can show [2] that $(\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle) = \bigwedge_{y \in Y} B_1(y) \leftrightarrow B_2(y)$. Therefore, measuring similarity of formal concepts via extents A_i coincides with measuring similarity via intents B_i, corresponding to the duality of the extent/intent view.

Given a truth degree $a \in L$ (threshold specified by a user), consider the thresholded relation ${}^a{\approx}$ on $\mathcal{B}(X,Y,I)$ defined by

$$\big(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle\big) \in {}^a{\approx} \quad \text{iff} \quad \big(\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle\big) \geqslant a.$$

That is, ${}^a{\approx}$ is the (crisp) relation "being similar to degree at least a." ${}^a{\approx}$ is reflexive and symmetric, but need not be transitive (it is transitive if $\mathbf{L}$ satisfies $a \otimes b = a \wedge b$). Call a subset B of $\mathcal{B}(X,Y,I)$ a ${}^a{\approx}$-block if it is a maximal subset of $\mathcal{B}(X,Y,I)$ such that each two formal concepts from B are similar to degree at least a (the notion of a ${}^a{\approx}$-block generalizes that of an equivalence class: if ${}^a{\approx}$ is an equivalence relation, ${}^a{\approx}$-blocks are exactly the equivalence classes). Denote by $\mathcal{B}(X,Y,I)/{}^a{\approx}$ the collection of all ${}^a{\approx}$-blocks.

It can be shown that ${}^a{\approx}$-blocks are special intervals in the concept lattice $\mathcal{B}(X,Y,I)$ [2,9]. In detail, for a formal concept $\langle A, B\rangle \in \mathcal{B}(X,Y,I)$, put

$$\langle A, B\rangle_a := \bigwedge\big\{\langle A', B'\rangle \mid \big(\langle A, B\rangle, \langle A', B'\rangle\big) \in {}^a{\approx}\big\}, \tag{1}$$

$$\langle A, B\rangle^a := \bigvee\big\{\langle A', B'\rangle \mid \big(\langle A, B\rangle, \langle A', B'\rangle\big) \in {}^a{\approx}\big\}. \tag{2}$$

That is, $\langle A, B\rangle_a$ and $\langle A, B\rangle^a$ are the infimum and the supremum of the set of all formal concepts which are similar to $\langle A, B\rangle$ to degree at least a. Then, ${}^a{\approx}$-blocks can be described as follows.

Lemma 1. *${}^a{\approx}$-blocks are exactly intervals of $\mathcal{B}(X,Y,I)$ of the form $[\langle A, B\rangle_a, (\langle A, B\rangle_a)^a]$, i.e.*

$$\mathcal{B}(X,Y,I)/{}^a{\approx} = \big\{\big[\langle A, B\rangle_a, \big(\langle A, B\rangle_a\big)^a\big] \mid \langle A, B\rangle \in \mathcal{B}(X,Y,I)\big\}.$$

Note that an interval with lower bound $\langle A_1, B_1\rangle$ and upper bound $\langle A_2, B_2\rangle$ is the subset $[\langle A_1, B_1\rangle, \langle A_2, B_2\rangle] = \{\langle A, B\rangle \mid \langle A_1, B_1\rangle \leqslant \langle A, B\rangle \leqslant \langle A_2, B_2\rangle\}$.

Now, define a partial order $\preccurlyeq$ on blocks of $\mathcal{B}(X,Y,I)/{}^a{\approx}$ by

$$[c_1, c_2] \preccurlyeq [d_1, d_2] \quad \text{iff} \quad c_1 \leqslant d_1 \quad (\text{iff } c_2 \leqslant d_2),$$

where $[c_1, c_2], [d_1, d_2] \in \mathcal{B}(X,Y,I)/{}^a{\approx}$ ($c_i \leqslant d_i$ denotes that in $\mathcal{B}(X,Y,I)$, c_i is a subconcept of d_i). Then we have

Theorem 2. *$\mathcal{B}(X,Y,I)/{}^a{\approx}$ equipped with $\preccurlyeq$ is a partially ordered set which is a complete lattice, the so-called factor lattice of $\mathcal{B}(X,Y,I)$ by similarity $\approx$ and a threshold a.*

Elements of $\mathcal{B}(X,Y,I)/{}^a{\approx}$ can be seen as similarity-based granules of formal concepts from $\mathcal{B}(X,Y,I)$. $\mathcal{B}(X,Y,I)/{}^a{\approx}$ thus provides a granular view on (a possibly large) $\mathcal{B}(X,Y,I)$. Note also that if ${}^a{\approx}$ is transitive then it is a congruence relation on $\mathcal{B}(X,Y,I)$ and $\mathcal{B}(X,Y,I)/{}^a{\approx}$ is the usual factor lattice modulo a congruence.

We now present an illustrative example. Consider the data table in Table 1. X contains nine objects (Mercury, ..., Pluto), Y contains four attributes ("size small," ..., "near to sun"). The corresponding concept lattice is depicted in Fig. 1.

Table 1
Data table with fuzzy attributes

	Size		From sun	
	small	large	far	near
Mercury (Me)	1	0	0	1
Venus (V)	1	0	0	1
Earth (E)	1	0	0	1
Mars (Ma)	1	0	$\frac{1}{2}$	1
Jupiter (J)	0	1	1	$\frac{1}{2}$
Saturn (S)	0	1	1	$\frac{1}{2}$
Uranus (U)	$\frac{1}{2}$	$\frac{1}{2}$	1	0
Neptune (N)	$\frac{1}{2}$	$\frac{1}{2}$	1	0
Pluto (P)	1	0	1	0

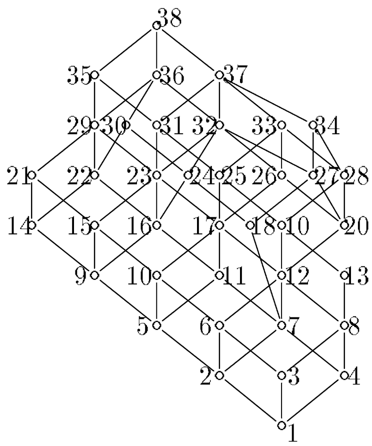

Fig. 1. Concept lattice $\mathcal{B}(X, Y, I)$ of data table from Table 1.

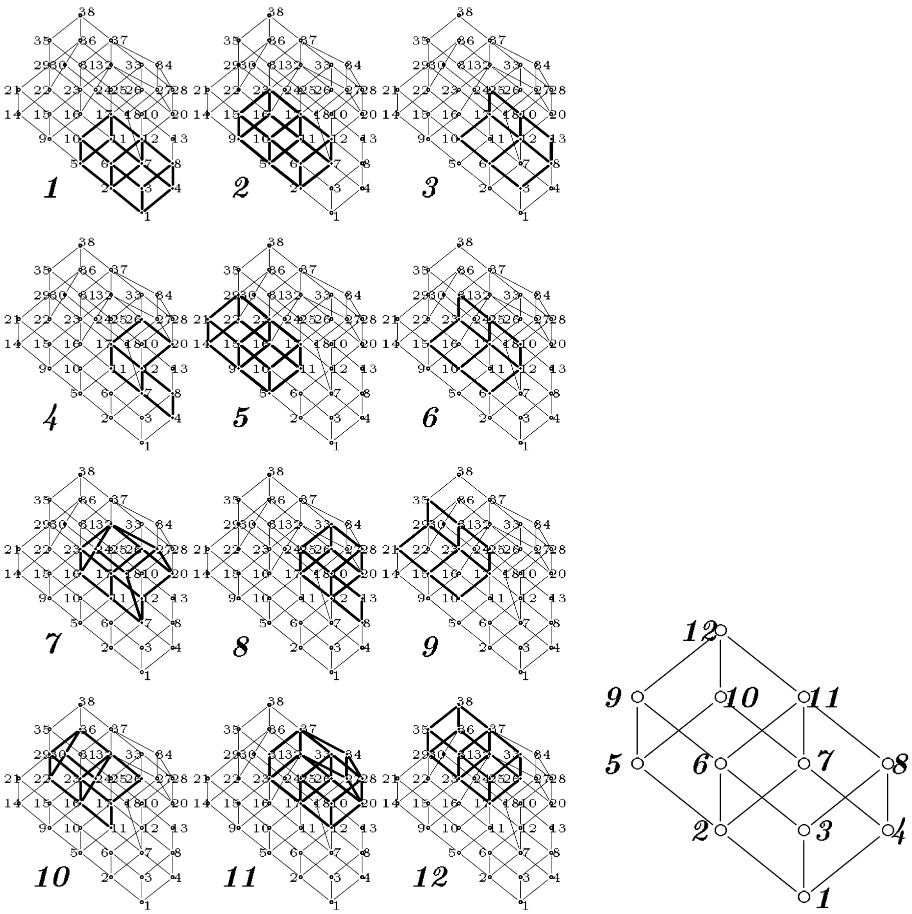

Fig. 2. $\frac{1}{2}\approx$-blocks of the concept lattice of Fig. 1 and the corresponding factor lattice $\mathcal{B}(X, Y, I)/\frac{1}{2}\approx$.

For $a = \frac{1}{2}$ there are twelve ${}^{1/2}\approx$-blocks and they are depicted in Fig. 2 (blocks are highlighted by solid lines) together with the corresponding factor lattice $\mathcal{B}(X, Y, I)/\frac{1}{2}\approx$.

3.2. Computing the factor lattice directly from input data

In order to compute $\mathcal{B}(X,Y,I)/{}^a\!\approx$ using definition and Lemma 1, one has (1) to compute the whole concept lattice $\mathcal{B}(X,Y,I)$ and then (2) to compute ${}^a\!\approx$-blocks on $\mathcal{B}(X,Y,I)$, which can be quite demanding. We are going to propose a way to compute $\mathcal{B}(X,Y,I)/{}^a\!\approx$ directly from input data $\langle X,Y,I\rangle$. We need some auxiliary results. For basic properties of concept lattices in fuzzy setting we refer to [2,5]. For a fuzzy set C in U and $a \in L$, the fuzzy sets $a \to C$ and $a \otimes C$ in U are defined by $(a \to C)(u) = a \to C(u)$ and $(a \otimes C)(u) = a \otimes C(u)$ for each $u \in U$. For fuzzy sets C, D in U, put $(C \approx D) = \bigwedge_{u\in U} C(u) \leftrightarrow D(u)$. Furthermore, we call a fuzzy set A in X an extent if there is a fuzzy set B in Y such that $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$ (dually, B is an intent if there is A with $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$).

Lemma 3. *If A is an extent then so is $a \to A$; if B is an intent then so is $a \to B$.*

Proof. We prove the assertion for extents. Let A be an extent, i.e. $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$ for some B. We have to show that $\langle a \to A, B'\rangle \in \mathcal{B}(X,Y,I)$. It suffices to show that $a \to A = (a \to A)^{\uparrow\downarrow}$ (since then $\langle a \to A, (a \to A)^{\uparrow}\rangle$ is a formal concept). Since $a \to A \subseteq (a \to A)^{\uparrow\downarrow}$ is always the case, we have to show $(a \to A)^{\uparrow\downarrow} \subseteq a \to A$ which holds iff $(a \to A)^{\uparrow\downarrow}(x) \leqslant a \to A(x)$ for each $x \in X$. Using adjointness, the latter is equivalent to $a \leqslant (a \to A)^{\uparrow\downarrow}(x) \to A(x)$. Since

$$(a \to A)^{\uparrow\downarrow}(x) \to A(x) \geqslant \bigwedge_{x\in X} (a \to A)^{\uparrow\downarrow}(x) \leftrightarrow A(x) = \big((a \to A)^{\uparrow\downarrow} \approx A\big),$$

it suffices to show $a \leqslant ((a \to A)^{\uparrow\downarrow} \approx A)$. First, we have $a \leqslant ((a \to A) \approx A)$. Indeed, from $a \leqslant ((a \to A(x)) \to A(x))$ and $a \leqslant (A(x) \to (a \to A(x)))$ for each $x \in X$ we have $a \leqslant ((a \to A(x)) \leftrightarrow A(x))$ for each $x \in X$, i.e. $a \leqslant \bigwedge_{x\in X}((a \to A(x)) \leftrightarrow A(x)) = ((a \to A) \approx A)$. Furthermore, since $(A_1 \approx A_2) \leqslant (A_1^{\uparrow} \approx A_2^{\uparrow})$ and $(B_1 \approx B_2) \leqslant (B_1^{\downarrow} \approx B_2^{\downarrow})$ for $A_1, A_2 \in L^X$ and $B_1, B_2 \in L^Y$ (see [2]), we have $(A_1 \approx A_2) \leqslant (A_1^{\uparrow} \approx A_2^{\uparrow}) \leqslant (A_1^{\uparrow\downarrow} \approx A_2^{\uparrow\downarrow})$. Putting this together, we get $a \leqslant ((a \to A) \approx A) \leqslant ((a \to A)^{\uparrow} \approx A^{\uparrow}) \leqslant ((a \to A)^{\uparrow\downarrow} \approx A^{\uparrow\downarrow})$, completing the proof. □

The next lemma shows that for a formal concept $\langle A,B\rangle$, $\langle A,B\rangle_a$ and $\langle A,B\rangle^a$, defined by (1) and (2) as infimum and supremum of all formal concepts similar to $\langle A,B\rangle$ to degree at least a, can be computed from $\langle A,B\rangle$ directly.

Lemma 4. *For $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$, we have*

(a) $\langle A,B\rangle_a = \langle (a \otimes A)^{\uparrow\downarrow}, a \to B\rangle$ *and*
(b) $\langle A,B\rangle^a = \langle (a \to A), (a \otimes B)^{\downarrow\uparrow}\rangle$.

Proof. Due to duality we verify only (a). The assertion follows from the following claims.

(a1) $(a \otimes A)^{\uparrow\downarrow}$ is an extent of a formal concept $\langle (a \otimes A)^{\uparrow\downarrow}, D\rangle$ which is similar to $\langle A,B\rangle$ to degree at least a;
(a2) if $\langle C,F\rangle$ is a formal concept similar to $\langle A,B\rangle$ to degree at least a then $\langle (a \otimes A)^{\uparrow\downarrow}, D\rangle \leqslant \langle C,F\rangle$;
(a3) $a \to B$ is an intent of a concept c which is similar to $\langle A,B\rangle$ to degree at least a;
(a4) if $\langle C,F\rangle$ is a concept similar to $\langle A,B\rangle$ to degree at least a then for c from (a3) we have $c \leqslant \langle C,F\rangle$.

Indeed, from (a1) and (a2) we get that $\langle (a \otimes A)^{\uparrow\downarrow}, D\rangle$ is the least formal concept similar to $\langle A,B\rangle$ to degree at least a. Therefore, $\langle A,B\rangle_a = \langle (a \otimes A)^{\uparrow\downarrow}, D\rangle$. Then, (a3) and (a4) yield that $a \to B$ is an intent of the least formal concept similar to $\langle A,B\rangle$ to degree at least a, i.e. $a \to B = D$. We now verify (a1)–(a4).

(a1): We have $a \leqslant ((a \otimes A) \approx A) \leqslant ((a \otimes A)^{\uparrow} \approx A^{\uparrow}) \leqslant ((a \otimes A)^{\uparrow\downarrow} \approx A^{\uparrow\downarrow}) = ((a \otimes A)^{\uparrow\downarrow} \approx A)$ since A is an extent.

(a2): If $a \leqslant (A \approx C)$ then using adjointness, we get $a \otimes A \subseteq C$ from which we have $(a \otimes A)^{\uparrow\downarrow} \subseteq C^{\uparrow\downarrow} = C$, proving (a2).

(a3): By Lemma 3, $a \to B$ is an intent. Using adjointness we easily get $a \leqslant (B \approx a \to B) = (\langle A,B\rangle \approx c)$.

(a4): We need to show $F \subseteq a \to B$. Since $a \leqslant (\langle A,B\rangle \approx \langle C,F\rangle) = (B \approx F)$, adjointness gives $a \otimes F \subseteq B$ and then $F \subseteq a \to B$. The proof is complete. □

Thus we have $(\langle A, B\rangle_a)^a = \langle a \to (a \otimes A)^{\uparrow\downarrow}, (a \otimes (a \to B))^{\downarrow\uparrow}\rangle$.

Lemma 5. *For $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ we have $\langle A, B\rangle_a = (((\langle A, B\rangle_a)^a)_a$.*

Proof. First we show that for every $c, d \in \mathcal{B}(X, Y, I)$ we have (1) $c \leqslant d$ implies $c_a \leqslant d_a$, (2) $c \leqslant d$ implies $c^a \leqslant d^a$, (3) $c \leqslant (c_a)^a$, (4) $c \geqslant (c^a)_a$. (1): Recall that $c_a = \bigwedge\{e \in \mathcal{B}(X, Y, I) \mid \langle c, e\rangle \in {}^a{\approx}\}$. We need to show that if $\langle d, f\rangle \in {}^a{\approx}$ then $c_a \leqslant f$. Thus suppose $\langle d, f\rangle \in {}^a{\approx}$. From $\langle c, c\rangle \in {}^a{\approx}$ and from the fact that ${}^a{\approx}$ is a tolerance relation compatible with lattice operations on $\mathcal{B}(X, Y, I)$ we get $\langle c, c \wedge f\rangle = \langle c \wedge d, c \wedge f\rangle \in {}^a{\approx}$. Now, since c_a is the infimum of all e such that $\langle c, e\rangle \in {}^a{\approx}$, we have $c_a \leqslant c \wedge f$ and since $c \wedge f \leqslant f$, we get $c_a \leqslant f$, proving (1). (2) can be proved analogously. (3) and (4) are obvious.

Now, let $c = \langle A, B\rangle$. By (3), $c \leqslant (c_a)^a$ and so $c_a \leqslant ((c_a)^a)_a$ by (1). Applying (4) to c_a we get $c_a \geqslant ((c_a)^a)_a$, proving $c_a = ((c_a)^a)_a$. $\square$

By Lemma 5, if a ${}^a{\approx}$-block $[c_1, c_2]$ is generated by $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$, i.e. $c_1 = \langle A, B\rangle_a$, $c_2 = (\langle A, B\rangle_a)^a$, then it is also generated by c_2, i.e. $c_1 = (c_2)_a$ and $c_2 = ((c_2)_a)^a$. Therefore, ${}^a{\approx}$-blocks $[c_1, c_2]$ are uniquely given by their suprema c_2. Moreover, since each formal concept $c_2 = \langle A, B\rangle$ is uniquely given by A (namely, $B = A^{\uparrow}$), ${}^a{\approx}$-blocks are uniquely given by extents of their suprema. Denote the set of all extents of suprema of ${}^a{\approx}$-blocks by $\mathrm{ESB}(a)$, i.e.

$$\mathrm{ESB}(a) = \bigl\{A \in L^X \bigm| \langle A, B\rangle \in \mathcal{B}(X, Y, I) \text{ and } \bigl[\langle A, B\rangle_a, \langle A, B\rangle\bigr] \in \mathcal{B}(X, Y, I)/{}^a{\approx}\bigr\}.$$

Before presenting the main result, let us recall that a fuzzy closure operator in a set X [3] is a mapping $C : A \to C(A)$ satisfying $A \subseteq C(A)$, $S(A_1, A_2) \leqslant S(C(A_1), C(A_2))$, and $C(A) = C(C(A))$, for any $A, A_1, A_2 \in L^X$. A fixed point of C is any fuzzy set A in X such that $A = C(A)$. Denote by $\mathrm{fix}(C)$ the set of all fixed points of C, i.e.

$$\mathrm{fix}(C) = \bigl\{A \in L^X \bigm| A = C(A)\bigr\}.$$

Theorem 6. *Given input data $\langle X, Y, I\rangle$ and a threshold $a \in L$, a mapping C_a sending a fuzzy set A in X to a fuzzy set $a \to (a \otimes A)^{\uparrow\downarrow}$ in X is a fuzzy closure operator in X for which* $\mathrm{fix}(C_a) = \mathrm{ESB}(a)$.

Proof. First, we verify that C_a is a fuzzy closure operator. $A \subseteq C_a(A)$ means $A \subseteq a \to (a \otimes A)^{\uparrow\downarrow}$ which is equivalent (by adjointness) to $a \otimes A \subseteq (a \otimes A)^{\uparrow\downarrow}$ which is true since $E \subseteq E^{\uparrow\downarrow}$ is always the case. We showed $A \subseteq C_a(A)$. $S(A_1, A_2) \leqslant S(C_a(A_1), C_a(A_2))$: Since for $D_1, D_2 \in L^U$, $S(D_1, D_2) \leqslant S(a \otimes D_1, a \otimes D_2)$ and $S(D_1, D_2) \leqslant S(a \to D_1, a \to D_2)$, see [4], we have

$$\begin{aligned} S(A_1, A_2) &\leqslant S(a \otimes A_1, a \otimes A_2) \leqslant S\bigl((a \otimes A_1)^{\uparrow\downarrow}, (a \otimes A_2)^{\uparrow\downarrow}\bigr) \leqslant S\bigl(a \to (a \otimes A_1)^{\uparrow\downarrow}, a \to (a \otimes A_2)^{\uparrow\downarrow}\bigr) \\ &= S\bigl(C_a(A_1), C_a(A_2)\bigr). \end{aligned}$$

To verify $C_a(A) = C_a(C_a(A))$, suppose first that A is an extent. Then, by Lemma 4, $C_a(A)$ is the extent of $(\langle A, A^{\uparrow}\rangle_a)^a$. In order to show $C_a(A) = C_a(C_a(A))$, we thus have to check $(\langle A, A^{\uparrow}\rangle_a)^a = ((((\langle A, A^{\uparrow}\rangle_a)^a)_a)^a$ which is true due to Lemma 5. If A is not an extent, the assertion follows from the fact that $C_a(A) = C_a(A^{\uparrow\downarrow})$, the fact that $A^{\uparrow\downarrow}$ is an extent and the previous claim. We thus need to check $C_a(A) = C_a(A^{\uparrow\downarrow})$. We have $a \leqslant (A \approx a \otimes A) \leqslant (A^{\uparrow\downarrow} \approx (a \otimes A)^{\uparrow\downarrow})$. So, $A^{\uparrow\downarrow}$ is similar to $(a \otimes A)^{\uparrow\downarrow}$ to degree at least a, whence $a \to (a \otimes A)^{\uparrow\downarrow} \supseteq A^{\uparrow\downarrow}$ since by Lemma 4, $a \to (a \otimes A)^{\uparrow\downarrow}$ is the greatest one which is similar to $(a \otimes A)^{\uparrow\downarrow}$ to degree at least a. In fact, in order to apply Lemma 4, A needs to be an extent. However, going through the proof, one can see that $(a \otimes A)^{\uparrow\downarrow}$ is the extent of the least formal concept which is similar to A to degree at least a even for an arbitrary fuzzy set A (not necessarily an extent). Therefore, the claim of Lemma 4 can be safely used in our case. We therefore have $A \subseteq A^{\uparrow\downarrow} \subseteq a \to (a \otimes A)^{\uparrow\downarrow}$ and since $a \otimes (a \to b) \leqslant b$, we get

$$(a \otimes A)^{\uparrow\downarrow} \subseteq \bigl(a \otimes A^{\uparrow\downarrow}\bigr)^{\uparrow\downarrow} \subseteq \bigl(a \otimes \bigl(a \to (a \otimes A)^{\uparrow\downarrow}\bigr)\bigr)^{\uparrow\downarrow} \subseteq \bigl((a \otimes A)^{\uparrow\downarrow}\bigr)^{\uparrow\downarrow} = (a \otimes A)^{\uparrow\downarrow}.$$

This proves $(a \otimes A)^{\uparrow\downarrow} = (a \otimes A^{\uparrow\downarrow})^{\uparrow\downarrow}$ and so $C_a(A) = a \to (a \otimes A)^{\uparrow\downarrow} = a \to (a \otimes A^{\uparrow\downarrow})^{\uparrow\downarrow} = C_a(A^{\uparrow\downarrow})$.

Second, we verify $\mathrm{fix}(C_a) = \mathrm{ESB}(a)$. Let $A \in \mathrm{fix}(C_a)$. By Lemma 1, the interval $[\langle A, A^{\uparrow}\rangle_a, (\langle A, A^{\uparrow}\rangle_a)^a]$ is a ${}^a{\approx}$-block, and by Lemma 4, $(\langle A, A^{\uparrow}\rangle_a)^a = \langle a \to (a \otimes A)^{\uparrow\downarrow}, \ldots\rangle$. Since $A = C_a(A) = a \to (a \otimes A)^{\uparrow\downarrow}$, A is the extent of a supremum of a block, i.e. $A \in \mathrm{ESB}(a)$. Conversely, let $A \in \mathrm{ESB}(a)$. Then $[\langle A, A^{\uparrow}\rangle_a, \langle A, A^{\uparrow}\rangle]$ is an ${}^a{\approx}$-block and so $(\langle A, A^{\uparrow}\rangle_a)^a = \langle A, A^{\uparrow}\rangle$. Lemma 4 now gives $A = a \to (a \otimes A)^{\uparrow\downarrow}$, i.e. $A = C_a(A)$ verifying $A \in \mathrm{fix}(C_a)$. □

Therefore, A is the extent of some formal concept c_2 which is the supremum of some ${}^a{\approx}$-block $[c_1, c_2] \in \mathcal{B}(X, Y, I)/{}^a{\approx}$ if and only if A is a fixed point of C_a. By Theorem 6 and the above considerations, going through $\mathrm{fix}(C_a)$ and computing for each $A \in \mathrm{fix}(C_a)$ the corresponding $[\langle A, A^{\uparrow}\rangle_a, \langle A, A^{\uparrow}\rangle] = [\langle (a \otimes A)^{\uparrow\downarrow}, a \to A^{\uparrow}\rangle, \langle A, A^{\uparrow}\rangle]$ generates all ${}^a{\approx}$-blocks of $\mathcal{B}(X, Y, I)/{}^a{\approx}$. Strictly speaking, we do not generate the ${}^a{\approx}$-blocks $[c_1, c_2] \in \mathcal{B}(X, Y, I)/{}^a{\approx}$ but only their boundary formal concepts $c_1, c_2 \in \mathcal{B}(X, Y, I)$. This is, however, in accordance with the purpose of the factorization of $\mathcal{B}(X, Y, I)$: We are looking for a granular view which is more concise than $\mathcal{B}(X, Y, I)$ itself.

The problem of computing $\mathcal{B}(X, Y, I)/{}^a{\approx}$ thus reduces to the problem of computing $\mathrm{fix}(C_a)$. To this end, we can use the algorithm described in [6]. The algorithm is an extension of the Ganter's algorithm generating all fixed points of an (ordinary) closure operator (see [9]) and generates all fixed points of a fuzzy closure operator C in a lexicographic order. Note that the algorithm in [6] is formulated in terms of the fuzzy closure operator ${}^{\uparrow\downarrow}$ (i.e. sending A to $A^{\uparrow\downarrow}$). But since each fuzzy closure operator is of the form of ${}^{\uparrow\downarrow}$, there is no loss of generality involved. We now briefly recall the algorithm from [6].

Let X and L be finite. Suppose $X = \{1, 2, \dots, n\}$ and $L = \{0 = a_1, a_2, \dots, a_k = 1\}$ such that if $a_i \leqslant a_j$ in $\mathbf{L}$ then $i \leqslant j$ (i.e. the ordering of elements of L by indices extends their ordering in $\mathbf{L}$). For $i, r \in \{1, \dots, n\}$, $j, s \in \{1, \dots, k\}$, put $(i, j) \leqslant (r, s)$ iff $i < r$, or $i = r$ and $j \geqslant s$. For $A \in L^X$, $(i, j) \in X \times \{1, \dots, k\}$, put $A \oplus (i, j) := C_a((A \cap \{1, 2, \dots, i-1\}) \cup \{a_j/i\})$. Here, $A \cap \{1, 2, \dots, i-1\}$ is the intersection of a fuzzy set A and the ordinary set $\{1, 2, \dots, i-1\}$, i.e. $(A \cap \{1, 2, \dots, i-1\})(x) = A(x)$ for $x < i$ and $(A \cap \{1, 2, \dots, i-1\})(x) = 0$ otherwise. Furthermore, for $A, B \in L^X$, put $A <_{(i,j)} B$ iff $A \cap \{1, \dots, i-1\} = B \cap \{1, \dots, i-1\}$ and $A(i) < B(i) = a_j$. Finally, put $A < B$ iff $A <_{(i,j)} B$ for some (i, j). Then $<$ is a total order on L^X and for each $A \in L^X$, the least fixed point $A^+ \in \mathrm{fix}(C_a)$ which is greater (w.r.t. $<$) than A is given by $A^+ = A \oplus (i, j)$ where (i, j) is the greatest one with $A <_{(i,j)} A \oplus (i, j)$ (see [6]). The algorithm for generating ${}^a{\approx}$-blocks which is based on this description of the successor operator ${}^+$ follows.

INPUT: $\langle X, Y, I\rangle$ (data table with fuzzy attributes), $a \in L$ (similarity threshold)
OUTPUT: $\mathcal{B}(X, Y, I)/{}^a{\approx}$ (collection of all ${}^a{\approx}$-blocks $[c_1, c_2]$)

```
/* Algorithm */
A := ∅
while A ≠ X do
   A := A⁺
   store([⟨(a ⊗ A)↑↓, a → A↑⟩, ⟨A, A↑⟩])
```

4. Examples and experiments

The aim of this section is to demonstrate experimentally the effect of reduction of size of a fuzzy concept lattice by factorization by similarity, and the speed-up of our algorithm. By reduction of size of a fuzzy concept lattice given by a data table $\langle X, Y, I\rangle$ with fuzzy attributes and a user-specified threshold a, we mean the ratio

$$\frac{|\mathcal{B}(X, Y, I)/{}^a{\approx}|}{|\mathcal{B}(X, Y, I)|}$$

of the number $|\mathcal{B}(X, Y, I)/{}^a{\approx}|$ of elements of $\mathcal{B}(X, Y, I)/{}^a{\approx}$, i.e. the number of elements of the factor lattice, to the number $|\mathcal{B}(X, Y, I)|$ of elements of $\mathcal{B}(X, Y, I)$, i.e. the number of elements of the original lattice. By a speed-up we mean the ratio of the time for computing the factor lattice $\mathcal{B}(X, Y, I)/{}^a{\approx}$ by a naive algorithm to the time for computing $\mathcal{B}(X, Y, I)/{}^a{\approx}$ by our algorithm. By "our algorithm" we mean the algorithm described in the end of Section 3. By "naive algorithm" we mean computing $\mathcal{B}(X, Y, I)/{}^a{\approx}$ by first generating $\mathcal{B}(X, Y, I)$ (by a polynomial time-delay algorithm from [6]) and subsequently generating the ${}^a{\approx}$-blocks by producing $[\langle A, B\rangle_a, (\langle A, B\rangle_a)^a]$.

Example 7. Consider the data table depicted in Table 2. The data table contains countries (objects from X) and some of their economic characteristics (attributes from Y). The values of the characteristics are scaled to interval [0, 1] so that the characteristics can be considered as fuzzy attributes.

Table 3 summarizes the results when using Łukasiewicz fuzzy logical operations and threshold values $a = 0.2, 0.4, 0.6, 0.8$. The whole concept lattice $\mathcal{B}(X, Y, I)$ contains 774 formal concepts.

The example demonstrates that smaller thresholds lead to both larger size reduction and speed-up. Furthermore, we can see that the time needed for computing the factor lattice $\mathcal{B}(X, Y, I)/^a\!\approx$ is smaller than time for computing the original concept lattice $\mathcal{B}(X, Y, I)$.

Note also that since computing $\mathcal{B}(X, Y, I)$ using the polynomial time delay algorithm from [6] takes 2292 ms, most of the time consumed by the naive algorithm is spent on factorization. For instance, for $a = 0.2$, 8995 ms is consumed in total of which 2292 ms is spent on computing $\mathcal{B}(X, Y, I)$ and $6703 = 8895 - 2292$ ms is spent on factorization, i.e. on computing $\mathcal{B}(X, Y, I)/^a\!\approx$ from $\mathcal{B}(X, Y, I)$.

Figure 3 contains graphs depicting reduction $|\mathcal{B}(X, Y, I)/^a\!\approx|/|\mathcal{B}(X, Y, I)|$ and speed-up from Table 3.

Table 4 and Fig. 4 show the same characteristics when using the minimum-based fuzzy logical operations (instead of Łukasiewicz fuzzy logical operations).

Example 8. In this example, the input data table $\langle X, Y, I\rangle$ comes from samples of results of IPAQ questionnaire. The purpose of the IPAQ (International Physical Activity Questionnaire) is to monitor various attributes related to physical activity of a population. We used two data sets collected during a research program at the Faculty of Physical Culture, Palacký University, Olomouc. The objects from X are both men and women in the Czech Republic who entered the questionnaire. The attributes are selected IPAQ-attributes, possibly scaled to [0, 1] so that they can be considered as fuzzy attributes. The first sample consists of 1000 objects and 8 attributes; the results for minimum-based logical operations are depicted in Table 5. The second sample consists of 4318 objects and 8

Table 2
Data table with fuzzy attributes

	1	2	3	4	5	6	7
1 Czech	0.4	0.4	0.6	0.2	0.2	0.4	0.2
2 Hungary	0.4	1.0	0.4	0.0	0.0	0.4	0.2
3 Poland	0.2	1.0	1.0	0.0	0.0	0.0	0.0
4 Slovakia	0.2	0.6	1.0	0.0	0.2	0.2	0.2
5 Austria	1.0	0.0	0.2	0.2	0.2	1.0	1.0
6 France	1.0	0.0	0.6	0.4	0.4	0.6	0.6
7 Italy	1.0	0.2	0.6	0.0	0.2	0.6	0.4
8 Germany	1.0	0.0	0.6	0.2	0.2	1.0	0.6
9 UK	1.0	0.2	0.4	0.0	0.2	0.6	0.6
10 Japan	1.0	0.0	0.4	0.2	0.2	0.4	0.2
11 Canada	1.0	0.2	0.4	1.0	1.0	1.0	1.0
12 USA	1.0	0.2	0.4	1.0	1.0	0.2	0.4

Attributes: 1—High Gross Domestic Product per capita (USD), 2—High Consumer Price Index (1995 = 100), 3—High Unemployment Rate (percent—ILO), 4—High production of electricity per capita (kWh), 5—High energy consumption per capita (GJ), 6—High export per capita (USD), 7—High import per capita (USD).

Table 3
Łukasiewicz fuzzy logical connectives, $\mathcal{B}(X, Y, I)$ of data from Table 2

	0.2	0.4	0.6	0.8
size $\lvert\mathcal{B}(X, Y, I)/^a\!\approx\rvert$	8	57	193	423
size reduction	0.010	0.073	0.249	0.546
naive algorithm (ms)	8995	9463	8573	9646
our algorithm (ms)	23	214	383	1517
speed-up	391.09	44.22	22.38	6.36

$|\mathcal{B}(X, Y, I)| = 774$, time for computing $\mathcal{B}(X, Y, I) = 2292$ ms; table entries for thresholds $a = 0.2, 0.4, 0.6, 0.8$.

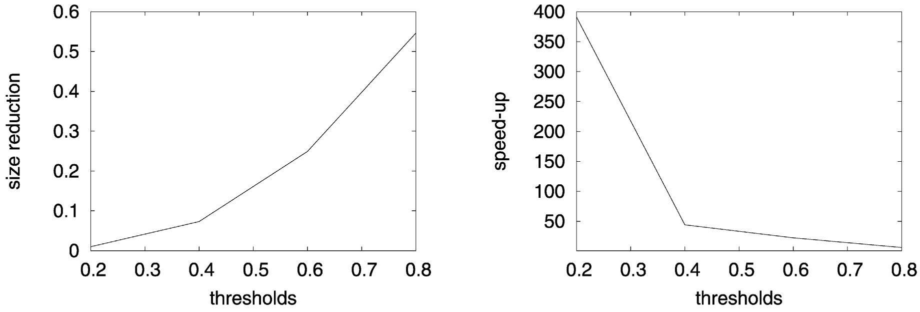

Fig. 3. Size reduction and speed-up from Table 3.

Table 4
Minimum-based fuzzy logical connectives, $\mathcal{B}(X, Y, I)$ of data from Table 2

	0.2	0.4	0.6	0.8
size $\lvert\mathcal{B}(X, Y, I)/^a\approx\rvert$	8	64	194	304
size reduction	0.026	0.210	0.638	1.000
naive algorithm (ms)	1830	1634	3787	4440
our algorithm (ms)	23	106	431	1568
speed-up	79.57	15.42	8.79	2.83

$\lvert\mathcal{B}(X, Y, I)\rvert = 304$, time for computing $\mathcal{B}(X, Y, I) = 341$ ms; table entries for thresholds $a = 0.2, 0.4, 0.6, 0.8$.

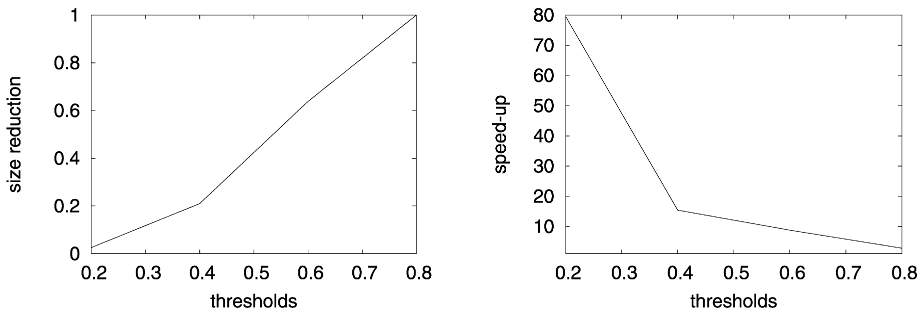

Fig. 4. Size reduction and speed-up from Table 4.

Table 5
$\lvert\mathcal{B}(X, Y, I)\rvert = 294$, time for computing $\mathcal{B}(X, Y, I) = 412$ sec

	0.25	0.5	0.75
size $\lvert\mathcal{B}(X, Y, I)/^a\approx\rvert$	63	130	208
size reduction	0.214	0.442	0.707
naive algorithm (sec)	436	444	444
our algorithm (sec)	121	232	322
speed-up	3.60	1.91	1.38

Table entries for thresholds $a = 0.25, 0.5, 0.75$.

attributes; the results for minimum-based logical operations are depicted in Table 6. Note that the differences between the speed-up in Tables 3 and 4, and in Tables 5 and 6 are mainly due to the differences in size reduction. This points out a natural property of our algorithm. Namely, the smaller the factor of size reduction, the larger the speed-up.

Table 6
$|\mathcal{B}(X,Y,I)| = 1095$, time for computing $\mathcal{B}(X,Y,I) = 376$ min

	0.25	0.5	0.75
size $\lvert\mathcal{B}(X,Y,I)/{}^{a}\!\approx\rvert$	142	416	697
size reduction	0.129	0.379	0.636
naive algorithm (min)	394	396	410
our algorithm (min)	75	175	284
speed-up	5.25	2.26	1.44

Table entries for thresholds $a = 0.25, 0.5, 0.75$.

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